Polytopics #28: Breaking Cundy's Deltahedra Record

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A **DELTAHEDRON** is a polyhedron all of whose faces are equilateral triangles, or "equits," as I call them for brevity. If we permit nonconvex figures or figures with intersecting faces, then the set of deltahedra is infinite, because we may join smaller deltahedra into larger "composite" deltahedra endlessly. But if we restrict ourselves just to *convex* deltahedra, then, perhaps surprisingly, their number shrinks to eight (a more manageable number).

The **eight convex deltahedra** were described as such in 1947 by H. Freudenthal and B. H. van der Waerden in their paper (in Dutch) "On an assertion of Euclid," in *Simon Stevin 25*:115–121, with an easy proof that the enumeration is complete. (But see also the Addendum at the end of this paper.) In 1952, H. Martyn Cundy published a short paper titled "Deltahedra" in the *Mathematical Gazette 36*: 263–266. Cundy followed up the Freudenthal & van der Waerden study by relaxing the convexity restriction but nevertheless restricting the nonconvex deltahedra to those whose vertices fall into a small number of symmetry-group equivalence classes (an attempt to keep the number of different deltahedra from exploding). In particular, he tabulated 17 putative acoptic (see next paragraph) nonconvex deltahedra with just two "kinds" (symmetry classes) of vertices. (Jonathan Bowers calls such polyhedra "biform"—"uniform" being polyhedra that have all regular faces and just one kind of vertex; see the 1954 monograph "Uniform Polyhedra" by H. S. M. Coxeter, M. Longuet-Higgins & J. C. P. Miller, in *Proceedings of the Royal Society of London*, Series A, *246*: 401–450, which describes and enumerates them all.) Norman W. Johnson later described all the convex polyhedra with regular faces (including as subsets the eight convex deltahedra and all the convex biform polyhedra) in his 1966 paper, "Convex Polyhedra with Regular Faces," in *The Canadian Journal of Mathematics 18*: 169–200. Cundy described a few biform star-deltahedra but did not attempt to enumerate them, although he did note the great icosahedron as the unique uniform star-deltahedron.

An *acoptic* polyhedron, so called by Branko Grünbaum, is one that may or may not be convex but whose faces nevertheless do not intersect (polyhedra with at least one pair of intersecting faces I distinguish with the term *star-polyhedra*). There can be no nonconvex acoptic uniform polyhedra. Nonconvex deltahedra and star-deltahedra, however, proliferate rapidly with increasing number of vertex-kinds, because one may join uniform deltahedra and pyramidal Johnson solids to any core biform deltahedron or uniform polyhedron with mainly equit faces, so for now it is practical to examine just those that are acoptic and biform. Later one may attempt to find all the triform ones (I suspect there are many hundreds), or to characterize the biform star-deltahedra (there are *infinitely many*, some of which are astonishingly intricate) and the nonconvex acoptic biform polyhedra that are not merely deltahedra but have other kinds of regular faces (there are also infinitely many).

I decided to see whether Cundy's proposed list of 17 acoptic nonconvex deltahedra with exactly two "kinds" of vertices was complete. Branko Grünbaum, in a recent email, made it clear that Cundy did *not* restrict himself in his paper to considering acoptic nonconvex deltahedra, in which case Cundy's list of 17 is far too short: As noted in the previous paragraph, there are infinitely many biform star-deltahedra (those in the infinite classes all have *n*-gonal antiprismatic symmetries, there being at least two such deltahedra for every $n>6$). To be fair, since Cundy published his paper in 1952 and Coxeter, Longuet-Higgins & Miller didn't publish their enumeration of the uniform polyhedra and star-polyhedra until 1954, Cundy would not necessarily have known of the various uniform star-polyhedra that may be augmented or excavated with octahedra or suitable equit-sided pyramids to yield interesting and in many instances stunningly intricate biform star-deltahedra. But even restricted to acoptic nonconvex biform figures, Cundy's list falls a bit short. Furthermore, three of his figures do not belong in his list: two are triform (they have three kinds of vertices) and the third is biform but not acoptic, because it has coincident edges and vertices. So Cundy actually tabulated only 14 acoptic biform deltahedra. There are at least eleven more deltahedra that belong in his table. This article describes the deltahedra that should have appeared there.

First, let me refine the definition of a *Cundy deltahedron* to be a polyhedron with the following characteristics (in order of restrictiveness):

#0: It is a legitimate, non-exotic polyhedron in Euclidean three-space (that is, its faces are filled planar polygons embedded in E[3], joined together exactly two per each edge, just as in a cardboard model), without coincident vertices or edges, and is not a compound of other such polyhedra.

#1: It is finite: The number of faces must be finite, so that the figure doesn't spread through three-space without bound, or fill a region of space densely with faces, and the faces themselves must also be bounded polygons that do not extend to infinity. The former condition eliminates infinite deltahedral towers, of which infinitely many acoptic uniform examples are conveniently provided by joining uniform n-gonal antiprisms together endlessly by their bases, for each $n>2$. Such towers, having the same arrangement of equits at every vertex, would qualify as regular "infinite polyhedra," except that they have edges of two different kinds.

#2: No two adjoining faces may lie in the same plane, although nonadjoining faces may be coplanar. This, in particular for deltahedra, eliminates several ways of augmenting tetrahedra onto a central octahedron, augmenting the cuboctahedron with pyramids on its square faces, and augmenting the icosidodecahedron with pyramids on its pentagonal faces. If coplanar adjoining faces were permitted, the number of convex deltahedra would become infinite, since the equit faces of any convex deltahedron may be subdivided into meshes of coplanar equits infinitely many ways.

#3: It is a deltahedron: All its faces are congruent equits; this constrains all the edges to have the same length, which we can set equal to 1 or 2, etc., as needed.

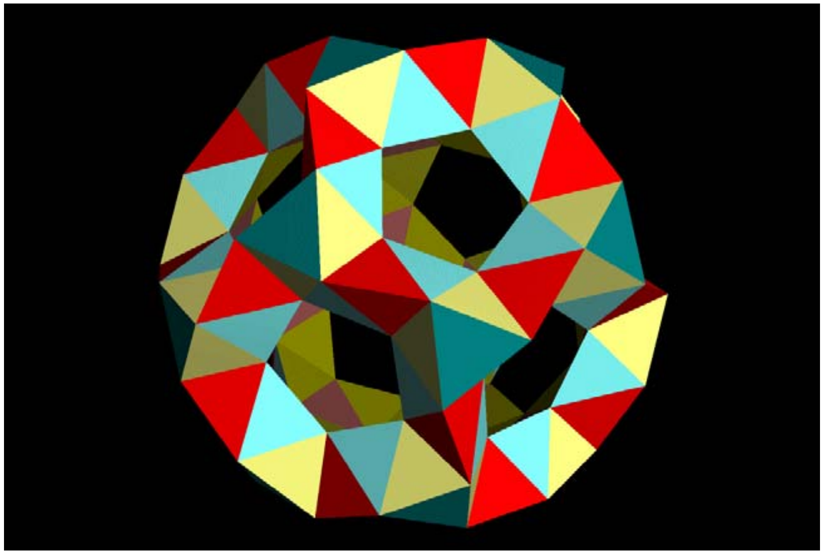
#4: It is nonconvex: It must have at least one reflex dihedral angle.

#5: It is acoptic (simple): No intersecting faces are allowed, nor are coincident edges and coincident vertices; so it is not a star-deltahedron, which must have at least one pair of intersecting faces (and almost always has many more). This eliminates the cuboctahedron whose square faces are excavated by pyramids, since the apices of the pyramids coincide at the center (this was figure #10 in Cundy's original list).

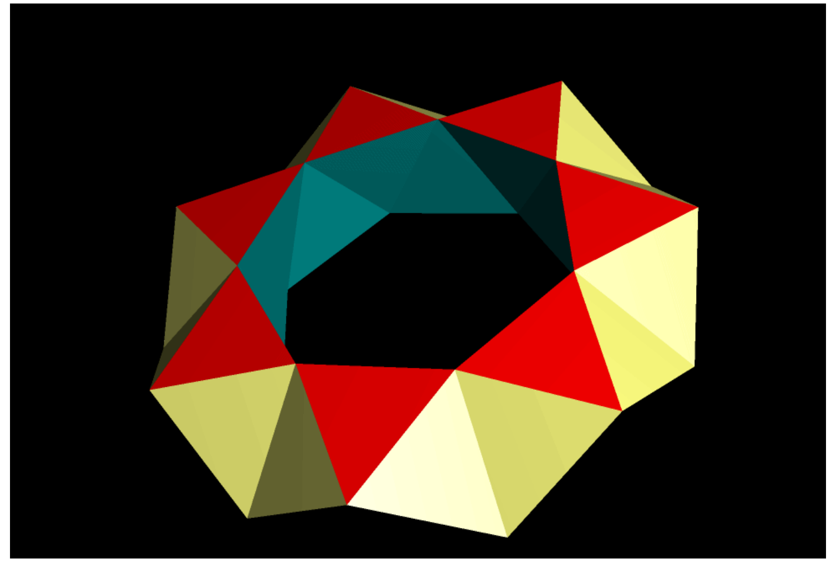
#6: It is biform: All the faces are regular polygons (trivially true for deltahedra), and the vertices fall into exactly two equivalence classes, so that all the vertices in either class are transitive on the polyhedron's symmetry group, but no symmetry of the polyhedron carries a vertex from either class into a vertex of the other class.

Condition #6 compels the vertices to lie on two concentric (possibly coincident) spheres centered at the polyhedron's center of symmetry (the point left invariant by any combination of the polyhedron's symmetry operations). This means that one may find all the Cundy deltahedra by symmetrically joining suitable "appendage" polyhedra, with mainly equit faces, to a uniform "core" polyhedron. It may be necessary to move the core's vertices symmetrically to new positions, perhaps onto a sphere with a different radius, to make the joins work. The vertices of the core polyhedron all continue to lie on one of two concentric spheres following the join, so the vertices of the appendage polyhedra that remain after the join must therefore all lie on the other concentric sphere.

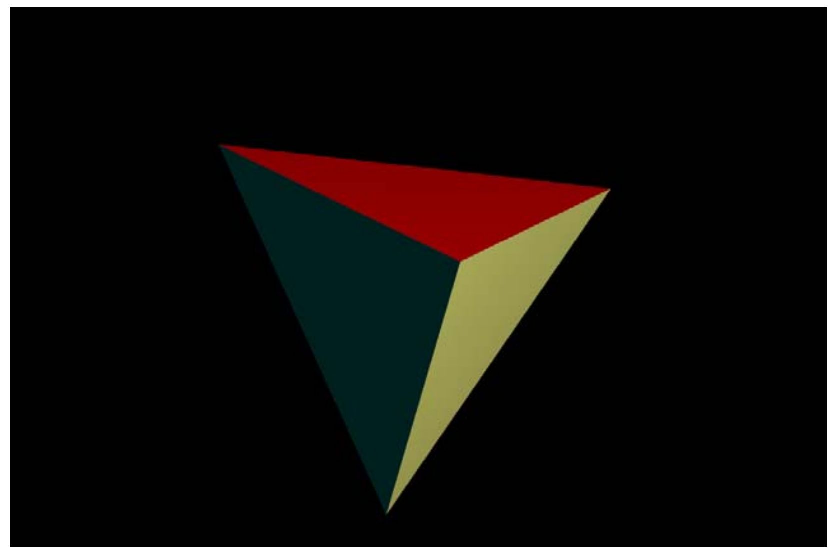
Having vertices on two concentric spheres also naturally brings up the possibility of a biform "cage" deltahedron. In a cage polyhedron, the center is surrounded by an inner polyhedron some of whose faces have been replaced by tunnels to the outer polyhedron, the whole figure forming what Bonnie Stewart has called a "toroid," or polyhedron of genus greater than zero. In the case of a cage deltahedron, the remaining faces of the inner and outer polyhedron, and the faces connecting the inner

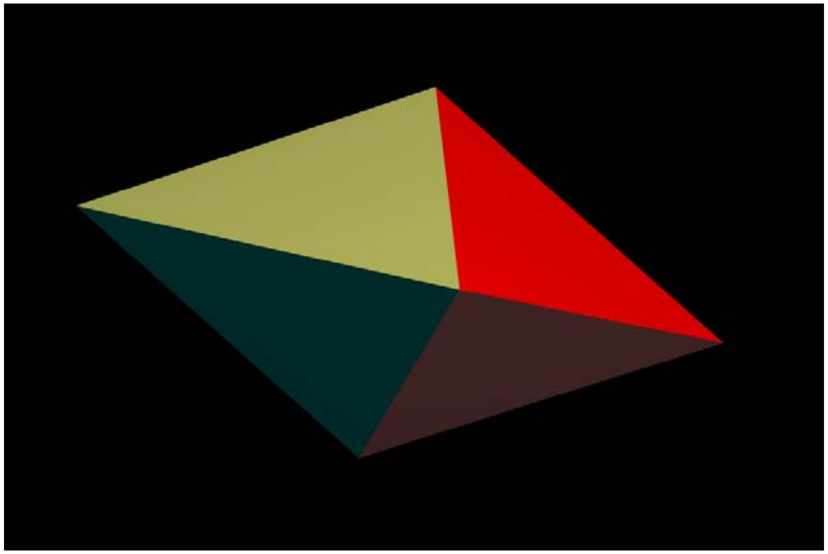


and outer polyhedron, must all be equits. The combination of being acoptic and biform and having all equit faces evidently overconstrains the problem, so that no acoptic biform cage deltahedra exist. Using the Great Stella program, I examined all the relevant possibilities and found that none leads to a true biform deltahedral cage. An interesting "near miss" is the case illustrated here, in which the inner and outer polyhedra start out as snub icosidodecahedra whose pentagons become the holes, connected by rings of triangles. Unfortunately, only 40 of the 280 triangles are equits; the rest are nearly equilateral triangles with interior angles close to but different from $\pi/3$. So the snub icosidodecahedral cage is only **quasi-biform**: it has two kinds of vertices, but not all of its faces are regular polygons. Quasi-biform n-gonal antiprismatic, icosahedral, and snub-cuboctahedral analogues of this kind of cage also exist. The antiprismatic "cages" are actually ring polyhedra, of which the heptagonal (n=7) example appears above right. In this near-miss deltahedron, all the triangles except the inner teal-colored ones are equits. The inner triangles are isosceles. The triangular (n=3) antiprismatic ring of this type is a regular icosahedron with opposite faces removed and replaced by the



triangles of a tall triangular antiprism, and the square (n=4) antiprismatic

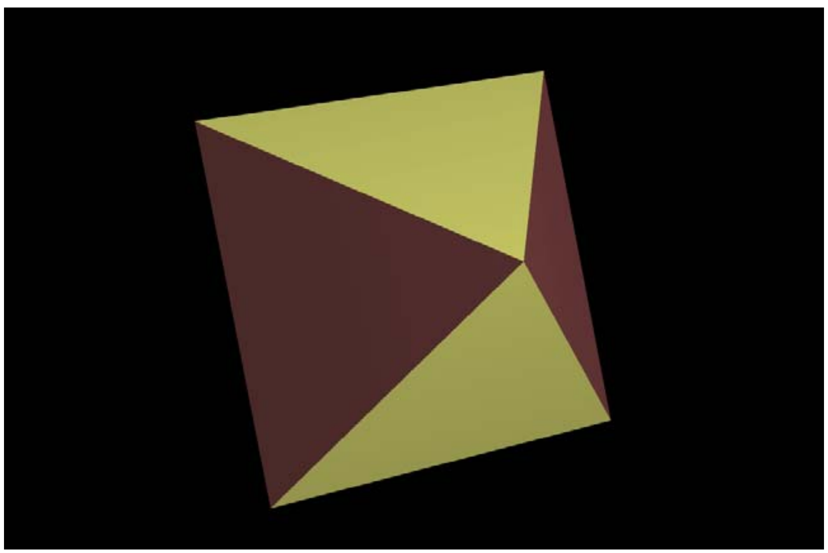


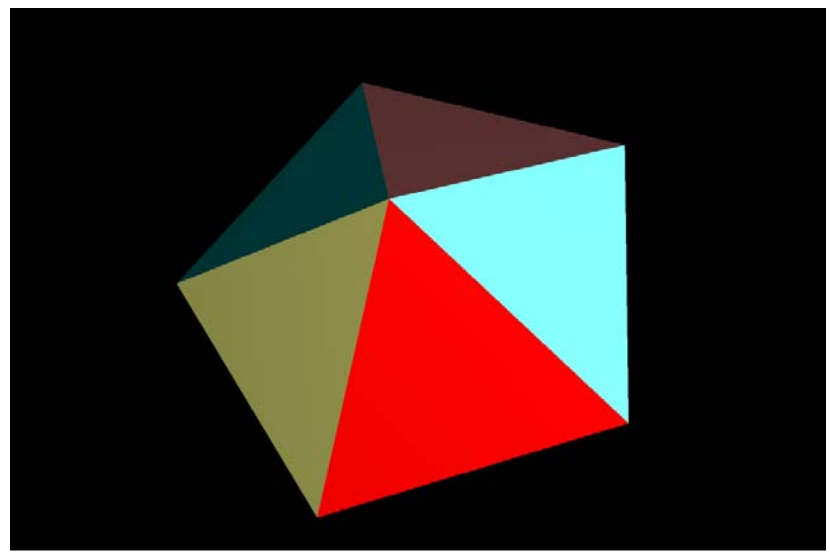


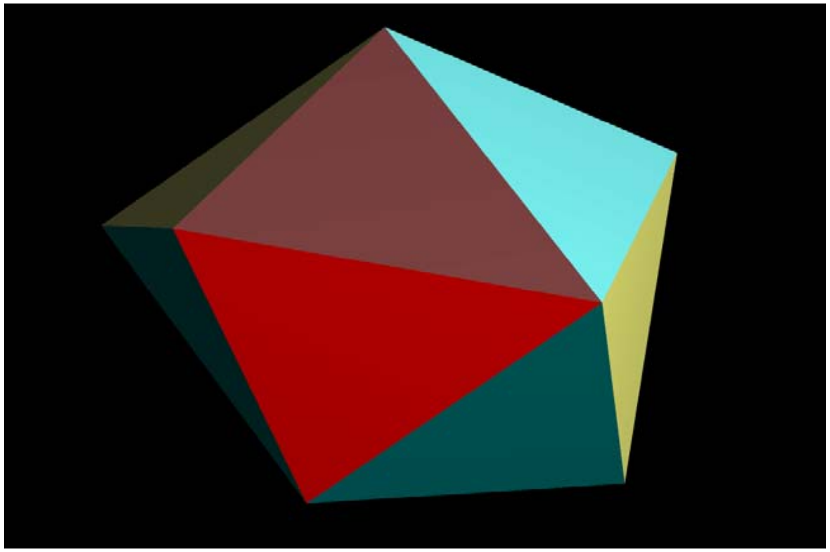
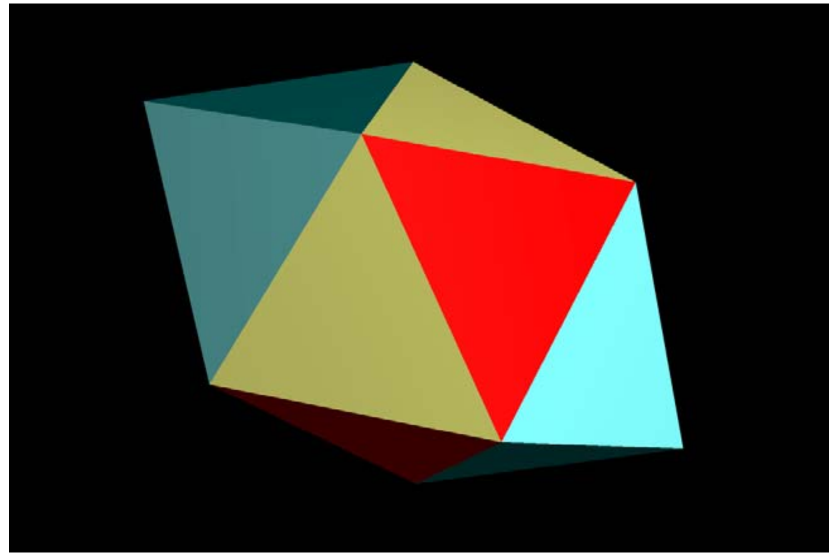
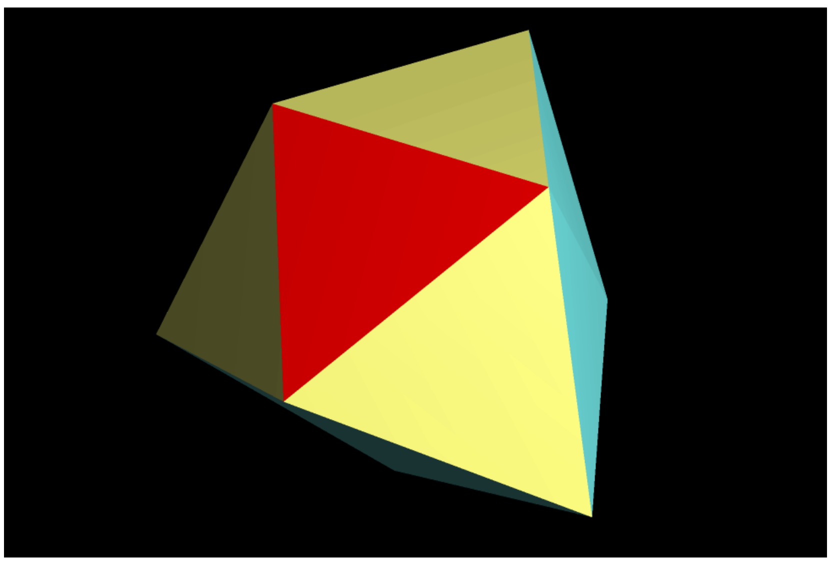
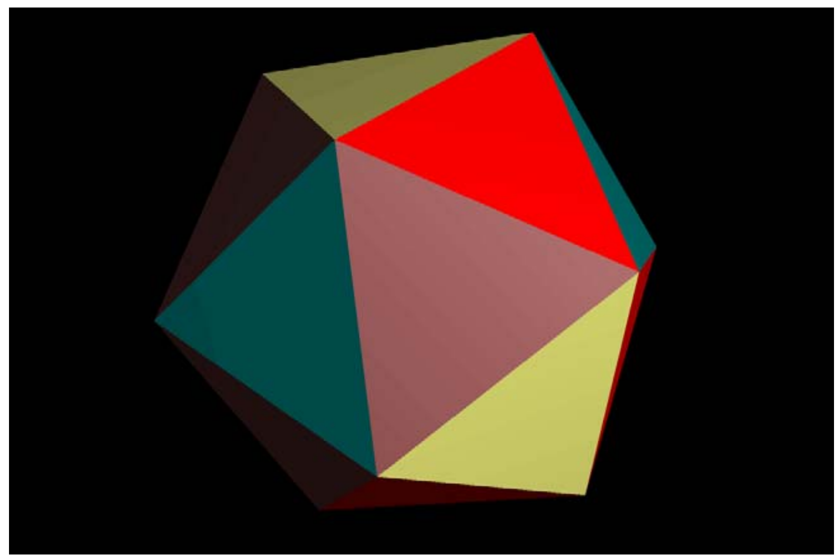
ring is the "snub square antiprism" Johnson solid with square faces removed and replaced by the triangles of a tall square antiprism. As *n* increases, the altitude of the removed antiprism diminishes, so that the inner triangles become less acute. The inner triangles become equits for a non-integer number between *n*=6 and *n*=7, so there is no integer *n* for which all the triangles in the ring are equits, and consequently no such Cundy deltahedral ring. (The *n*=7 ring is closest of all to being a Cundy deltahedron.) Since there thus presently seem to be no toroidal Cundy deltahedra, all Cundy deltahedra will satisfy Euler's formula V−E+F=2.

Before proceeding further, let me for completeness list and illustrate (on pp. 3–5) the eight convex deltahedra of Freudenthal & van der Waerden, in order of increasing number of faces. Three are regular (because they are uniform and have just one kind of regular polygon, namely the equit, for a face), the other five happen to be biform:

[1] **Regular tetrahedron**.
[2] **Triangular bipyramid** (or **monaugmented tetrahedron** in the





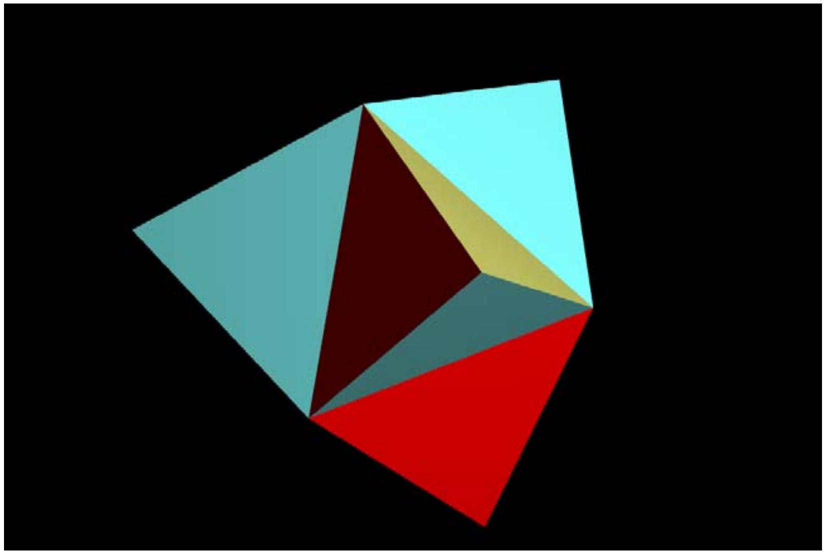





nomenclature developed below).
[3] **Regular octahedron** (also a **square bipyramid**, a **triangular antiprism**, and a **monaugmented square pyramid** in the nomenclature developed below).
[4] **Pentagonal bipyramid** (or **monaugmented pentagonal pyramid** in the nomenclature developed below).
[5] **Snub disphenoid** (or **dispheniated tetrahedron** in the nomenclature developed below).
[6] **Triaugmented triangular prism**.
[7] **Diaugmented square antiprism** (or **gyroelongated square bipyramid**).
[8] **Regular icosahedron** (which is also a **gyroelongated pentagonal bipyramid** and a **diaugmented pentagonal antiprism**).

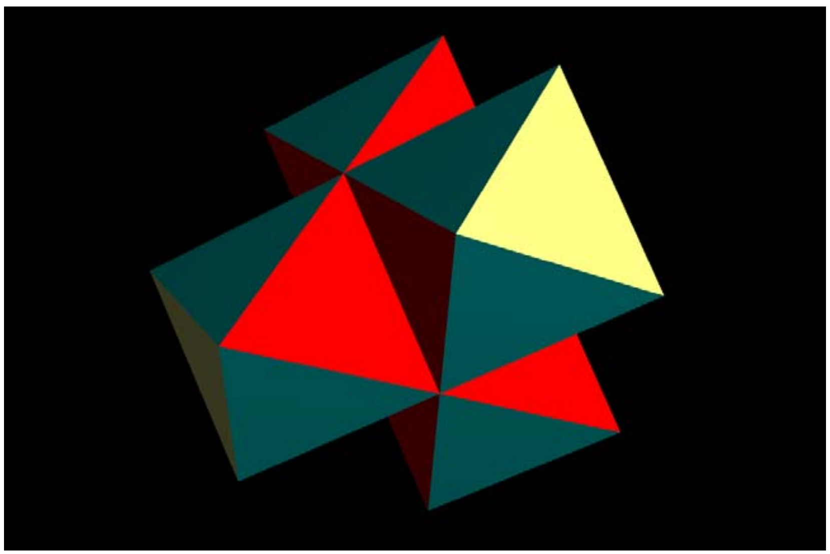
Of these, only [1] and [5] cannot be constructed directly by joining smaller regular, uniform, or Johnson polyhedra. [2] is a pair of regular tetrahedra joined at a common face; [3] is a pair of square pyramids joined at their common square face; [4] is a pair of pentagonal pyramids joined at their common pentagonal face; [6] is a triangular prism with square pyramids joined to its three square faces; [7] is a square antiprism with square pyramids joined to its two square faces; and among the several ways [8] may be assembled from smaller polyhedra is by joining two pentagonal pyramids to the bases of a pentagonal antiprism. This exemplifies the flavor of the operations Cundy used to construct the nonconvex acoptic deltahedra in his table. A convex deltahedron cannot have more than five equits at a vertex; the theoretical maximum of five at every vertex is attained by the regular icosahedron. This accounts for the small number of members of this subclass of deltahedra. Cundy's names for [5], [6], and [7] above are, respectively, **dodecadeltahedron**, **tetracaidecadeltahedron**, and **heccaidecadeltahedron**; see H. M. Cundy & A. P. Rollett, *Mathematical Models*, Oxford University Press, 1951, p. 136.

Anyway, criteria #0 through #6 seem to define a reasonably interesting collection of polyhedra. It is time to see what they are; their pictures appear on pp. 6–14, 15, and 17 in the order I have tabulated them. With the possibilities of ring and cage deltahedra already eliminated, we are essentially left with the operation of joining (that is, augmenting outwardly or excavating inwardly) one or more appendage polyhedra to a uniform core polyhedron, the appendages symmetrically joined to one or more faces of the core, whose vertices may or may not require relocation to accommodate the join. The core must be uniform because, *e.g.*, joining appendages to a biform core would result in at least a triform rather than a biform polyhedron. If the join is just to one face, then there can be no vertex relocation, because the core is convex (all uniform acoptic polyhedra are convex) and rigid. When the join covers a patch of two or more faces, then one or more edges between faces in the patch may be eliminated, which permits a certain amount of movement of the core vertices to make any non-equit triangles into equits. Convex deltahedron [5] above provides an example of this kind of join, wherein a tetrahedron is "spheniated" at two opposite edges. These "movable join" operations will become clearer below, as we construct these kinds of Cundy deltahedra.

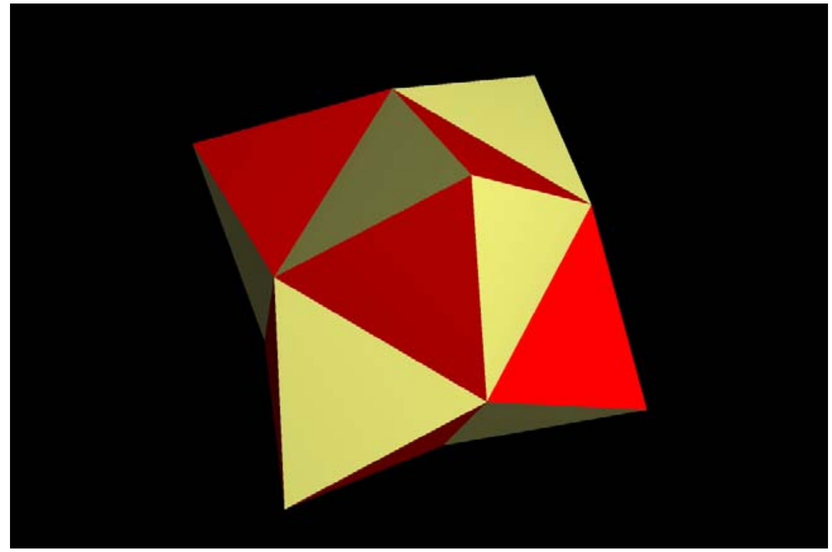
First, note that in augmenting or excavating a suitable uniform core



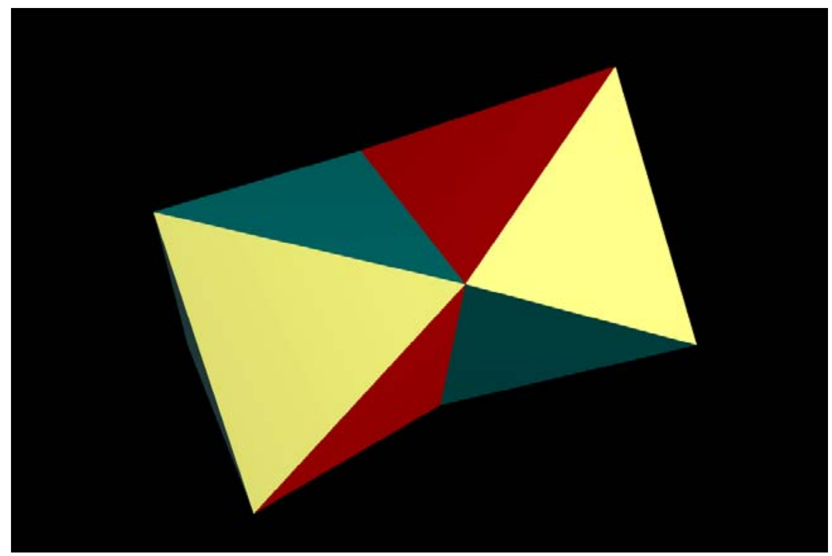
polyhedron, the augmentation or excavation must hit each core vertex identically, otherwise the core will automatically acquire a second kind of vertex and the resulting figure will immediately be cooked as non-biform. With regard to joining appendage polyhedra to a core at a single face, the only possible appendages are pyramids with equit lateral faces, in which case the base must be the join face, and the octahedron, in which case the join face on the core polyhedron must be an equit. Adding these appendages will add just a single kind of vertex to the core. All other potential appendages, such as the other deltahedra or Johnson solids with one non-equit face, have more than one plane of vertices above any potential join face. Hence the resulting deltahedron will automatically be cooked by the biform limitation. Also, the core uniform polyhedron cannot have any n-gons as faces for n>5, since there are no corresponding equit-sided pyramids to join to these faces. Indeed, the core polyhedron can have at most one kind of non-equit face, to all of which appropriate equit-



sided pyramids must be joined.

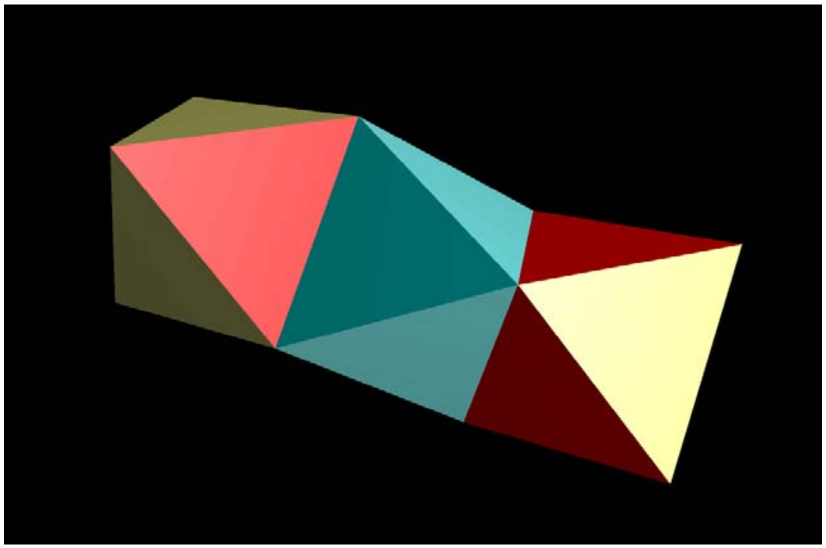


In naming the figures we find, I reserve the term “augmented” for a



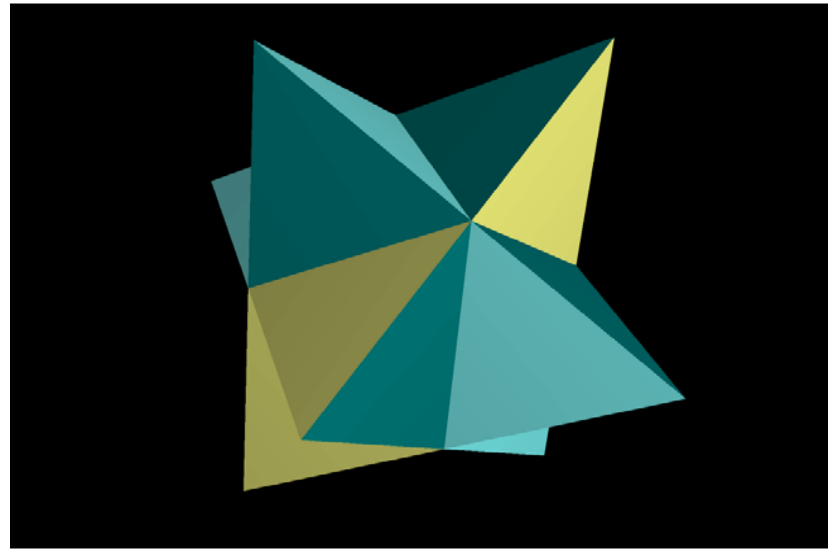
polyhedron to which pyramids have been joined with apices away from the center of symmetry, “excavated” for a polyhedron to which pyramids have been joined with apices toward the center of symmetry (that is, from which pyramids have been removed), and “gyraugmented” for a polyhedron to which octahedra have been joined with the face opposite the join face located farther from the center of symmetry. The octahedron is too big to create acoptic “gyrexcavated” biform polyhedra (polyhedra with the face opposite the join face located closer to the center of symmetry) with any candidate core polyhedron.

[1] **Tetraugmented tetrahedron**: A tetrahedron to which four more



tetrahedra have been joined, this figure can also serve as a net for a regular pentachoron. The symmetry group of this deltahedron is the full tetrahedral symmetry group [[3,3]]. In Cundy’s original “table of deltahedra with two kinds of vertex” it was #1. (8–18+12)

[2] **Tetragyraugmented tetrahedron**: A tetrahedron to which four octahedra have been joined, this figure can also serve as part of a net of a

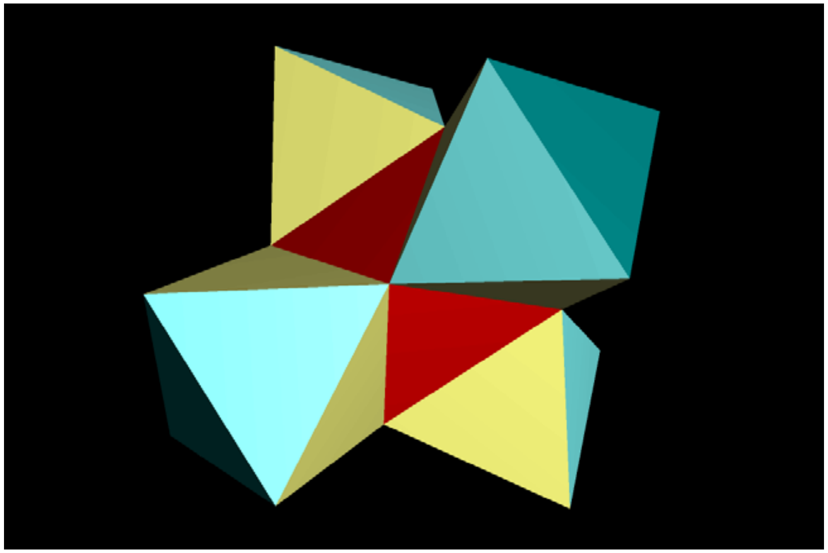


dispentachoron. As with [1], its symmetry group is [[3,3]]. It was #2 in Cundy’s original table. (16–42+28)

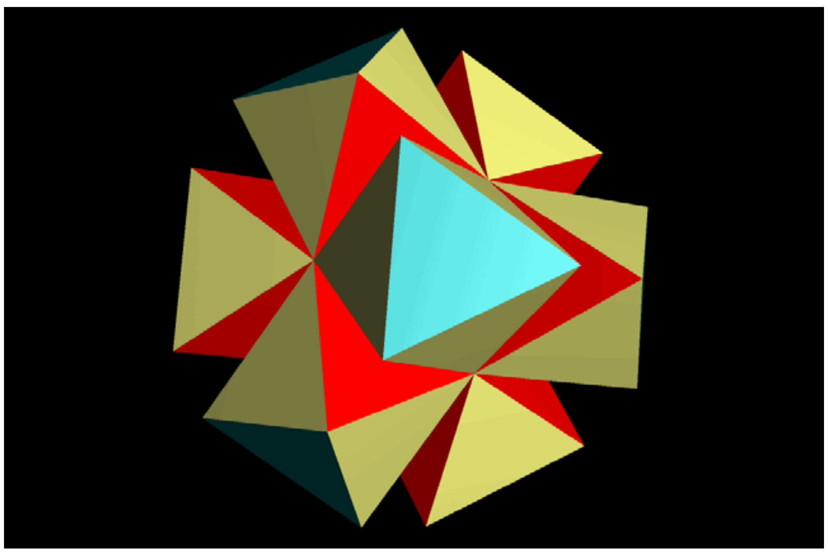
[3] **Hexaugmented cube**: A cube to which six square pyramids have been joined, this figure can also serve as a net of a Johnson cubic pyramid in E(4). Its symmetry group is the full octahedral group [[3,4]]. It was #7 in Cundy’s original table. (14–36+24)

[4] **Gyraugmented octahedron**: Comprising two octahedra joined at a common face, here either octahedron acts as an appendage to the other, so it may also be called “Siamese-twin octahedra.” The symmetry group of this deltahedron is the full triangular dihedral group [[2,3]]. It was absent from Cundy’s original table. (9–21+14)

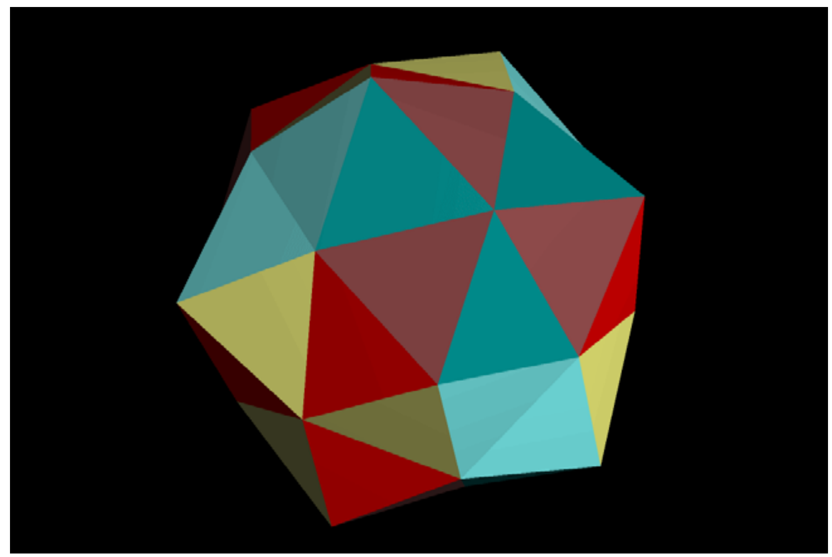
[5] **Digyraugmented octahedron**: A tower of three octahedra, this delathedron is an octahedron with two more octahedra joined to two opposite faces. Here the top and bottom octahedra act as appendages to the middle one, so it may also be called **“**Siamese-triplet octahedra.” Its

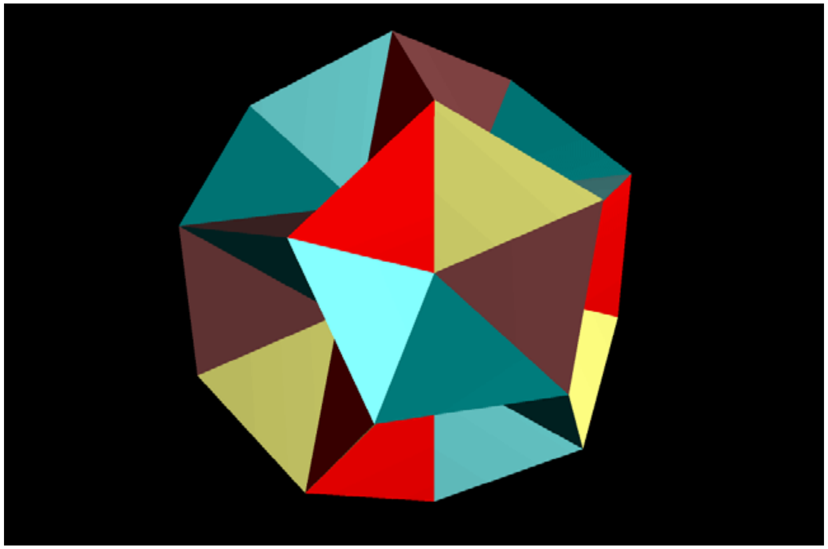


symmetry group is the full triangular antiprismatic group $[[2^+,6]]$. It was also absent from Cundy's original table. (12−30+20)



[6] **Octaugmented octahedron**: An octahedron to which eight tetrahedra have been adjoined, this figure can also serve as a net of the Johnson octahedral pyramid in E(4) — half of a regular hexadecachoron. This figure resembles the Stella Octangula (regular compound of two tetrahedra), but here the sets of coplanar equits are regarded as separate faces instead of being the externally visible parts of intersecting giant equits. The potential "tetraugmented octahedron" (an octahedron with tetrahedra joined to only half its faces) is simply a giant tetrahedron, because the lateral equits of the pyramids are coplanar by threes with the four adjacent unaugmented faces of the core octahedron. It is therefore excluded from the set of Cundy deltahedra. For similar reasons, the cuboctahedron cannot be used as a core polyhedron in this exercise — square pyramids either have their apices coincide at the center (if they are excavated) or have coplanar equits with the core equits (if they are augmented); this kind of coplanarity also prohibits the "diaugmented" and "hexaugmented" octahedra, which attempt to use the octahedron's

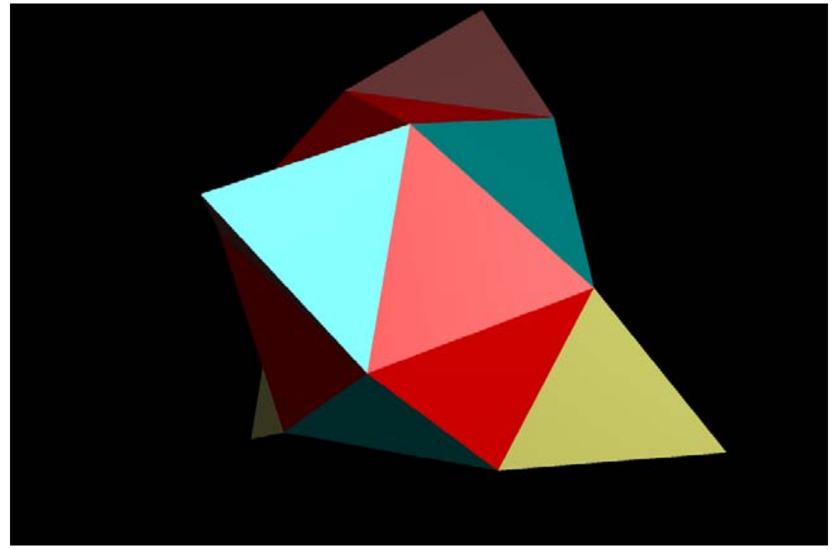




antiprismatic symmetry with augmenting tetrahedra, respectively at two opposite faces or at the other six faces. The octaugmented octahedron has the full octahedral symmetry group [[3.4]]. It was #6 in Cundy's original table. (14–36+12)

[7] **Tetragyraugmented octahedron**: An octahedron to which four octahedra have been adjoined in tetrahedral symmetry, this deltahedron can also serve as a net of five adjacent octahedra from the regular icositetrachoron. It has the full tetrahedral symmetry group [[3,3]], and was absent from Cundy's original table. Evidently Cundy overlooked the possibilities of augmenting, excavating, and gyraugmenting symmetric subsets of the faces of a core uniform polyhedron. Although we can gyraugment one, two, four, and all eight faces of an octahedron to obtain a biform deltahedron, we cannot obtain one by gyraugmenting a girdle of six octahedral faces: The resulting deltahedron is triform. (18–48+32)

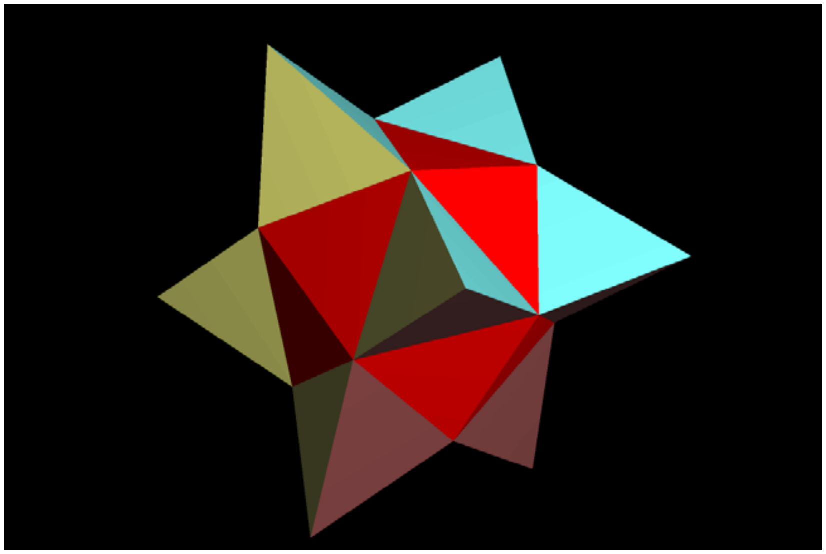
[8] **Octagyraugmented octahedron**: An octahedron to which eight octahedra have been adjoined, this figure can also serve as part (exactly 9/24) of a net of the regular icositetrachoron. It has the full octahedral

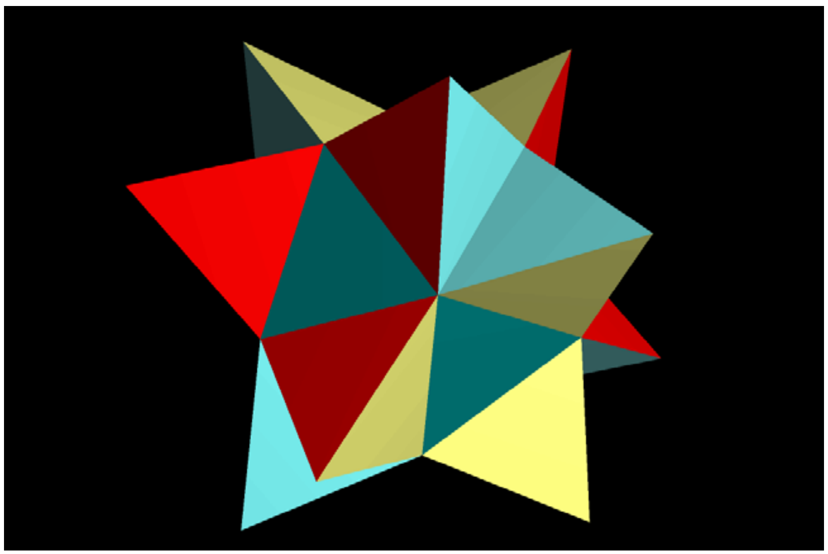


symmetry group [[3,4]], and was #3 in Cundy's original table. (30–84+56)

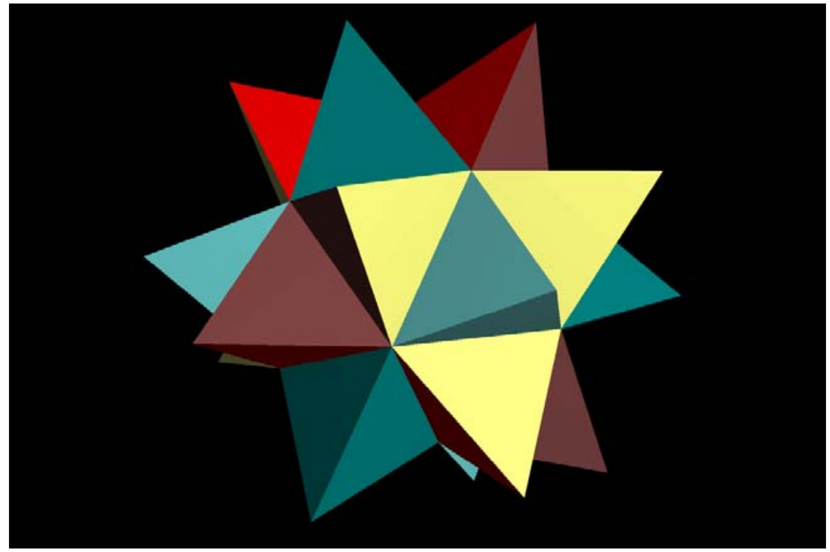
[9] **Dodecaugmented dodecahedron**: A regular dodecahedron to which twelve pentagonal pyramids have been joined, this will not fold up in E(4) as a net of a Johnson dodecahedral pyramid (because it doesn't exist). It has the full icosahedral symmetry group [[3,5]], and it was #8 in Cundy's original table. (32–90+60)

[10] **Dodekexcavated dodecahedron**: A regular dodecahedron from which twelve pentagonal pyramids have been removed. Since the E(4) Johnson dodecahedral pyramid does not exist, we may excavate Johnson pentagonal pyramids from a regular dodecahedron without fear of their apices crashing around the center. Also called a "dimpled dodecahedron," this deltahedron, like the dodecaugmented dodecahedron, has the full icosahedral symmetry group [[3,5]]. It was #9 in Cundy's original table. It is also an aggregation (or "stellation") of the regular icosahedron, since the equits fall into 20 coplanar sets of three. Considered as an icosahedron



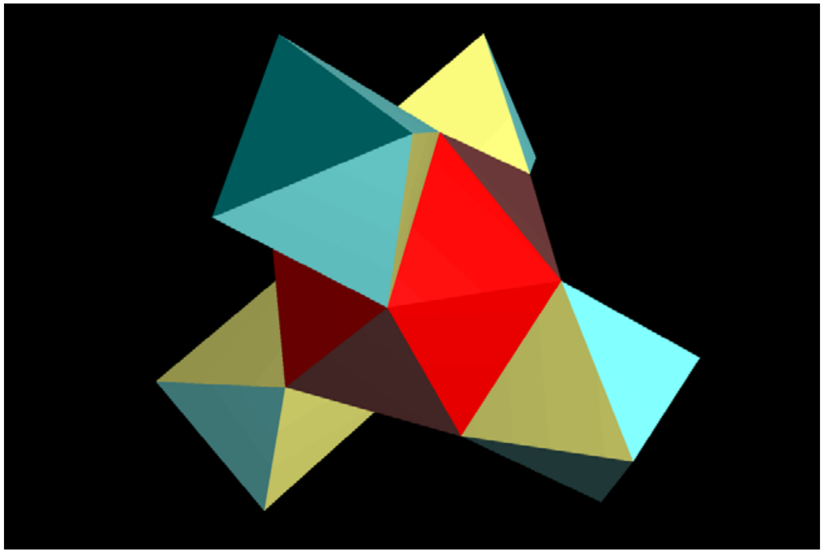


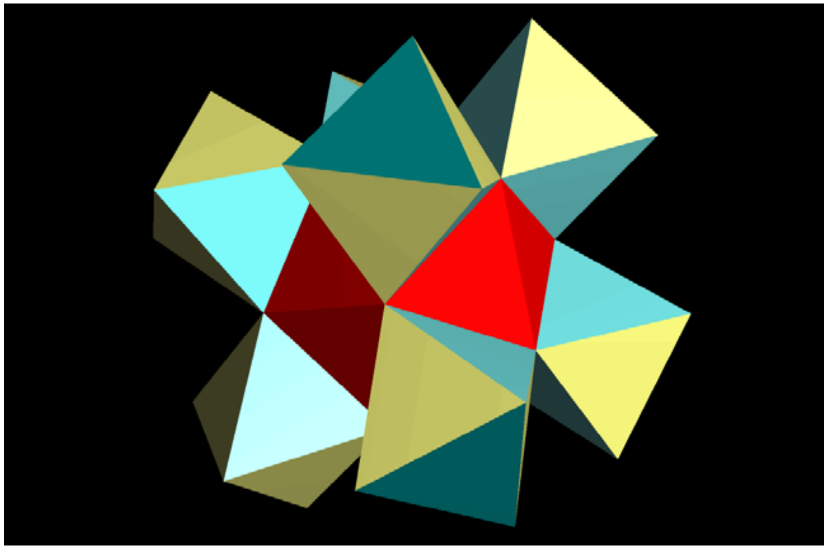
with 20 “propeller hexagons” as its faces (the “blades” of the propellers are the externally visible equits; the equit centers of the propellers form an entirely hidden regular icosahedron whose vertices are the centers of the



“dimples”; one set of three blades happens to be colored red in the illustration), this polyhedron, having the 20 vertices of the original dodecahedron, is self-dual and quasi-uniform. (32−90+60)

[11] **Tetraugmented icosahedron**: A regular icosahedron to which four tetrahedra have been joined at four tetrahedrally located faces, this figure has the tetrahedral rotational symmetry group [3,3] and is chiral — it and its relatives [12] and [13] below make use of the fact that an icosahedron is also a snub tetratetrahedron, and that the symmetry group [3,3] is a subgroup of index 5 of the icosahedral rotational symmetry group [3,5]. This deltahedron was absent from Cundy’s table. Altogether, we may augment an icosahedron on four, eight, twelve, or 20 faces to produce acoptic biform deltahedra. Alas, if we excavate the icosahedron in these ways, the resulting figures are not acoptic: The tetrahedra crash around the



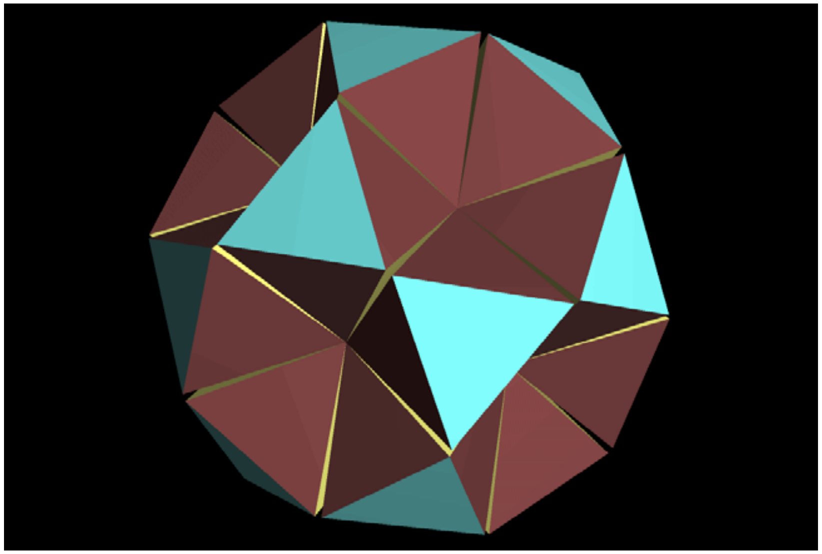


center, because the Johnson icosahedral pyramid exists in E(4).

(16–42+28)

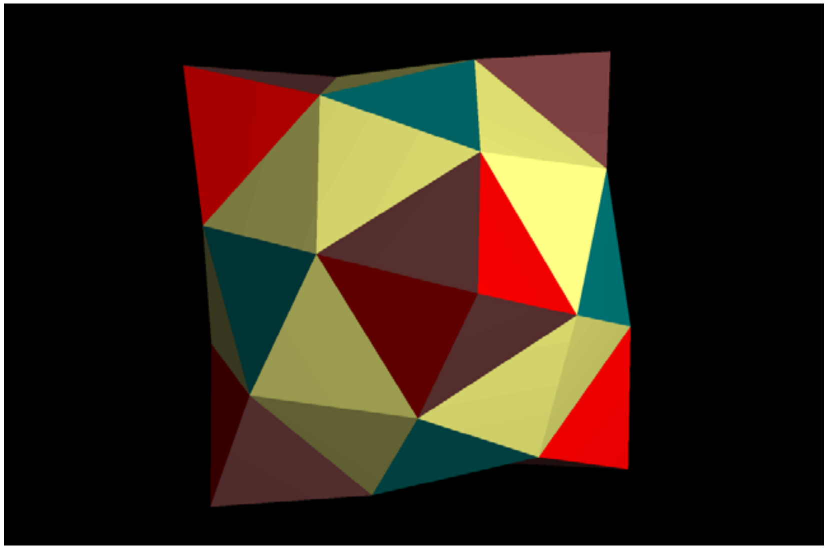
[12] **Octaugmented icosahedron**: A tetraugmented icosahedron to which four more tetrahedra have been joined, to the icosahedral faces opposite the first four appendages; this figure has the ionic (pyritohedral) symmetry group [3⁺,4]. This symmetry group is a subgroup of index 5 of the full icosahedral group [[3,5]]. (20–54+36)

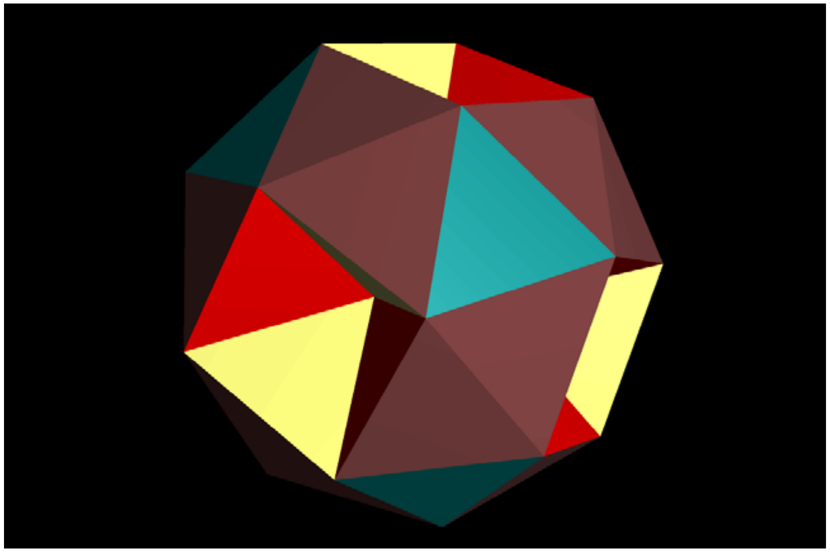
[13] **Dodecaugmented icosahedron**: A regular icosahedron to which



twelve tetrahedra have been joined, to the twelve faces not augmented to create [12], this figure likewise has the ionic (pyritohedral) symmetry group [3⁺,4]. (24–66+44)

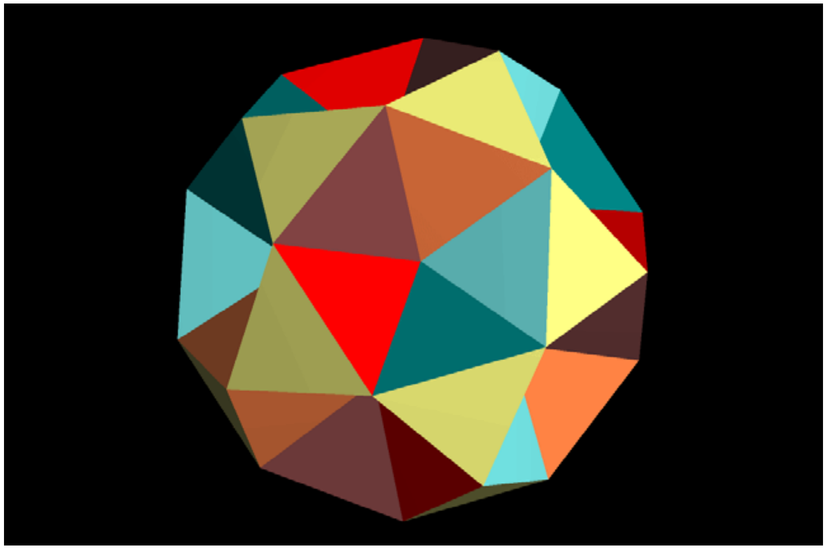
[14] **Icosaugmented icosahedron**: A regular icosahedron to which 20 tetrahedra have been joined, to all 20 of its faces, this figure can also serve as the net of a Johnson icosahedral pyramid in E(4)—a chip off the regular hexacosichoron. It has the full icosahedral symmetry group [[3,5]], and it



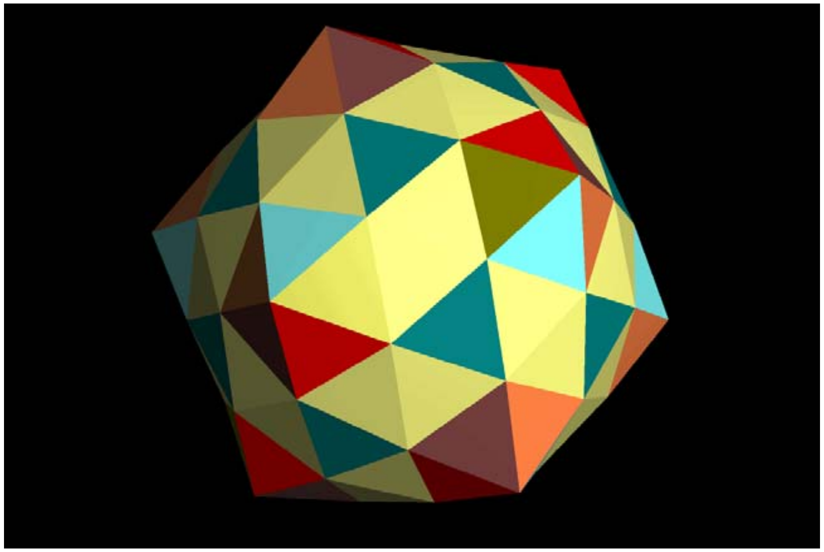


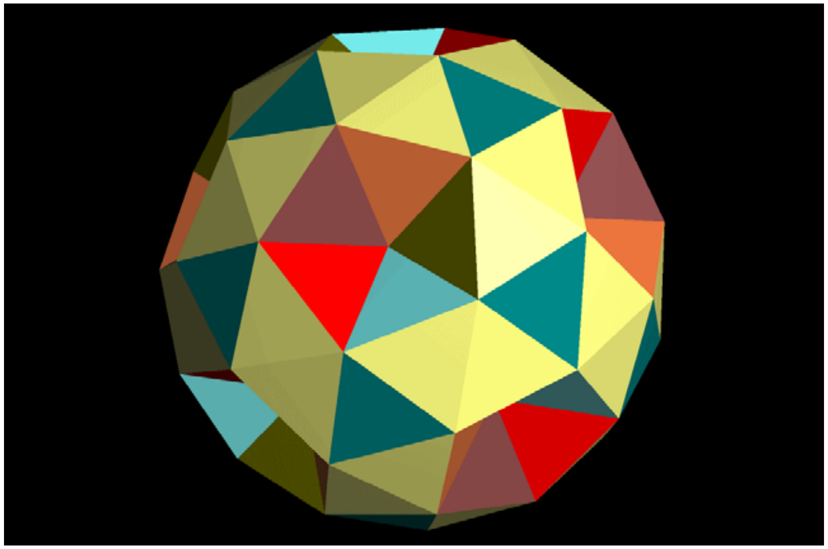
was #4 in Cundy's original table. (32−90+60)

[15] **Tetragyraugmented icosahedron**: A regular icosahedron to which four octahedra have been joined at four tetrahedrally located faces, this figure is also a very small part of the net of the rectified hexacosichoron in E(4). Like its augmented counterpart [11] above, it has the rotational tetrahedral symmetry group [3,3] and is chiral. Also like its counterpart, it was absent from Cundy's original tabulation. We can also gyraugment eight and all 20 faces of the icosahedron, but gyraugmenting twelve results in a triform deltahedron. (24−66+44)



[16] **Octagyraugmented icosahedron**: A tetragyraugmented icosahedron to which four more octahedra have been joined at the faces opposite the first four appendages, this figure has the ionic (pyritohedral) symmetry group $[3^+,4]$. Unfortunately, joining twelve octahedra to an icosahedron where we joined the tetrahedra for [12] above (at the red equits in the figure) produces a triform deltahedron (the pyritohedral group has 24





symmetries, but twelve octahedra place 36 vertices in the second sphere). Like [12] and [15], this deltahedron was absent from Cundy's original tabulation. (36–102+68)

[17] **Icosagyraugmented icosahedron**: A regular icosahedron to which 20 octahedra have been joined, to all 20 of its faces, this deltahedron can also serve as a small part of the net in E(4) of the icosahedral hexacosihecaton-icosachoron or rectified hexacosichoron, whose cells are 120 regular icosahedra and 600 regular octahedra. It has the full icosahedral symmetry group [[3,5]] and was #5 in Cundy's original table. In the illustration, the yellow equits of the octahedral appendages are barely visible, forming narrow dihedral crevasses. (72–210+140)

[18] **Hexaugmented snub cuboctahedron**: A snub cuboctahedron to which six square pyramids are joined; this figure has the octahedral rotational symmetry group [3,4] and is chiral. It and its excavated counterpart [19] are the only acoptic biform deltahedra with octahedral rotational symmetry. It was #14 on Cundy's original list. (30–84+56)

[19] **Hexexcavated snub cuboctahedron**: A snub cuboctahedron from which six square pyramids have been removed, this figure also has the octahedral rotational symmetry group [3,4] and is chiral. It was #15 on Cundy's original list. Great Stella confirms that the excavated square pyramids do not crash around the center of the figure. (30–84+56)

[20] **Dodekexcavated icosidodecahedron**: An icosidodecahedron from which twelve pentagonal pyramids have been removed, this figure has the full icosahedral symmetry group [[3,5]]. It was #13 in Cundy's priginal tabulation. Augmenting, rather than excavating, the icosidodecahedron renders the pyramid equits coplanar with the icosidodecahedron equits, so that the figure simply becomes a big icosahedron and violates the coplanarity condition for adjacent faces. Deltahedron [20] is almost the same as [17] above, but the latter has the narrow dihedral crevices that [20] does not. (42–120+80)

[21] **Dodecaugmented snub icosidodecahedron**: A snub icosidodeca-

hedron to which twelve pentagonal pyramids have been joined at its pentagonal faces, this deltahedron and its excavated counterpart [22] are the only acoptic biform deltahedra with the rotational icosahedral symmetry group [3.5] and are chiral. It was #16 in Cundy's original tabulation. (72–210+140)

[22] **Dodekexcavated snub icosidodecahedron**: A snub icosidodecahedron from which twelve pentagonal pyramids have been removed at its pentagonal faces. It was #17 (the last) in Cundy's original tabulation. See also the entry for [21]. (72–210+140)

Here let me break briefly from the tabulation to describe spheniation and ambiation, related operations used to construct the three remaining Cundy deltahedra. Those three deltahedra cannot be constructed simply by augmenting, excavating, gyraugmenting, or gyrexcavating a core polyhedron with appendage polyhedra the way deltahedra [1] through [22] were, or by joining together two or more simpler deltahedra or regular-faced polyhedra.

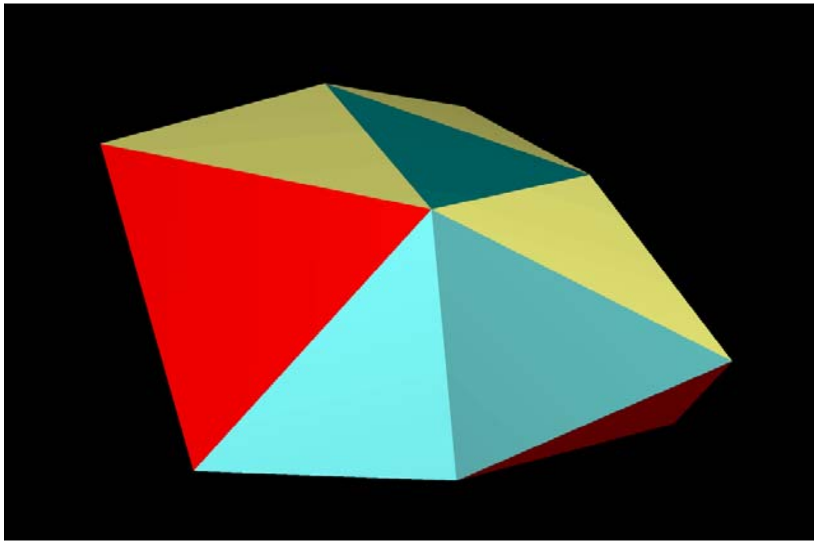
Whereas augmentation, excavation, gyraugmentation, and gyrexcavation join each appendage polyhedron to its own single core face, it is also possible to join a connected group of appendage polyhedra to a larger patch of faces. When just a single face is involved in a join, vertex movement and shape changes are unnecessary. But when two or more core faces are involved, the join, to succeed, may require moving the vertices of the core and/or changing the shapes of the appendage and/or core faces. Shape changes may take place because some edges of the patch are lost in the join, which allows the remaining faces to change shape and/or move to new positions. To maintain biformity, symmetrically repeated patches should cover most of the surface of the uniform core, hitting all its vertices identically. Then the appendages can be applied simultaneously to all the patch images, and the necessary vertex motions and face-shape changes will occur symmetrically.

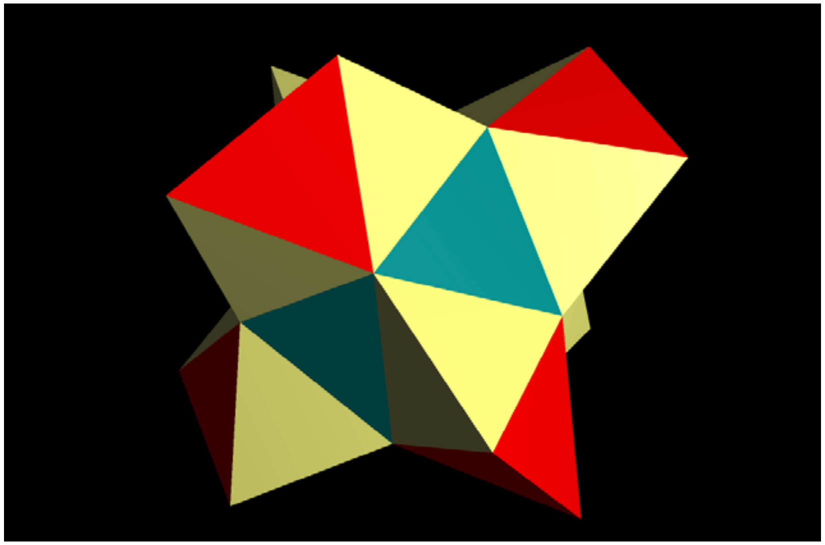
In **spheniation**, a patch comprises two adjacent faces. Each face is augmented or excavated, and the apices of the pyramids are then bridged by a pair of triangles across and above (or below) the edge at which the two patch faces adjoin. This makes a little wedge of the two triangles from one pyramid to the other, hence the name "spheniation." In the process of constructing spheniated deltahedra, it is typically necessary to change the shapes of the wedge triangles into equits, which in turn requires moving the vertices of the patch faces and the appended faces to accommodate those new shapes. Great Stella has a "spring model" function that changes the faces of an irregular-faced polyhedron into faces that are as regular as possible without sacrificing the polyhedron's topology. This function is usually sufficient to create a true deltahedron from an initial polyhedron whose faces are not all equits, assuming that that is possible.

The simplest instance of spheniation may be performed on the tetrahedron, whose faces fall into two patches of two equits each. Spheniating both patches simultaneously replaces each equit pair with a wedge of six equits, and the resulting deltahedron is none other than Freudenthal and van der Waerden's [5], the snub disphenoid. In my nomenclature, that figure is the **dispheniated tetrahedron**.

The lateral equits of an n-gonal antiprism fall into n congruent pairs that may be spheniated symmetrically around the figure. This infinite family of figures was recently discovered by Mason Green and Jim McNeill, and in particular the case n=3 results in a new Cundy deltahedron, [23], described below.

Ambiation is a generalization of spheniation to a patch of more than two faces. Specifically, the patch is a regular n-gon surrounded on all sides by n equits. Two such patches make up an n-gonal antiprism, for example; six such square patches and eight such equit patches may be found distributed symmetrically over the surface of a snub cuboctahedron; and twelve such pentagonal patches and 20 such equit patches occur on the surface of the snub icosidodecahedron. We join an n-gonal antiprism to the central n-gon of each patch (making a little table out of the n-gonal face, hence the name "ambiation"), then put a wedge of two triangles between each equit of the patch and the equit of the antiprism that adjoins it. Then we apply the Great Stella "spring model" function to the polyhedron, to regularize





the triangles. This adds a net total of $2n$ equits to the polyhedron per patch.

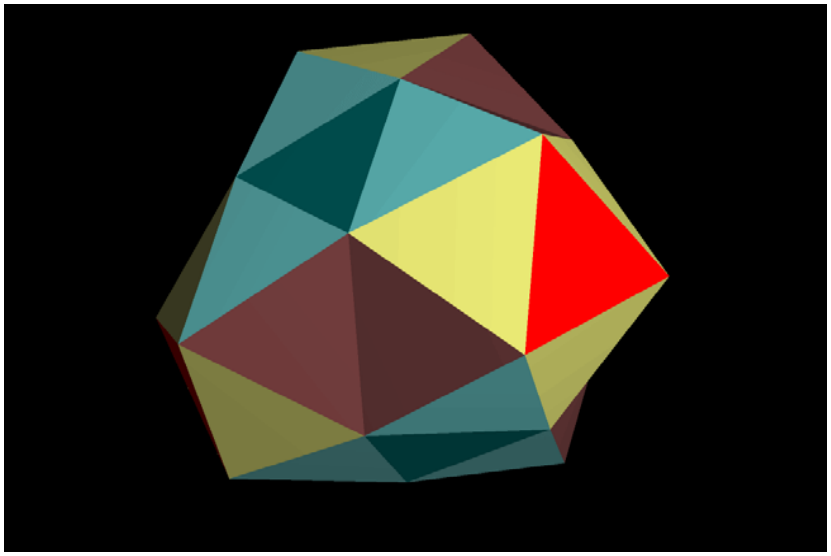
The astute reader will recognize spheniation as ambiation for the instance n=2.

The simplest instance of equit (trigonal) ambiation may be performed on the octahedron, which, being an equit antiprism, has a surface of two equit patches of four equits each. The resulting "diambiated octahedron" is none other than the regular icosahedron. The two four-equit patches ambiate into the two ten-equit patches that make up the icosahedron.

Diambiating the square antiprism at its two squares produces the Johnson polyhedron known as the snub square antiprism. Diambiating an n-gonal antiprism for n>4 produces a nonconvex biform polyhedron with the full n-gonal antiprismatic symmetry group $[[2^+,2n]]$. As n goes to infinity, the distance between the opposite n-gons of the **diambiated *n*-gonal antiprism** tends to 0. Note that this is a different operation from the symmetrical n-spheniation of an antiprism around its lateral equit faces described above. Replacing the two n-gons of a diambiated n-gonal antiprism with a quasi-uniform n-gonal antiprism of suitable altitude results in the quasi-biform ring polyhedra described earlier.

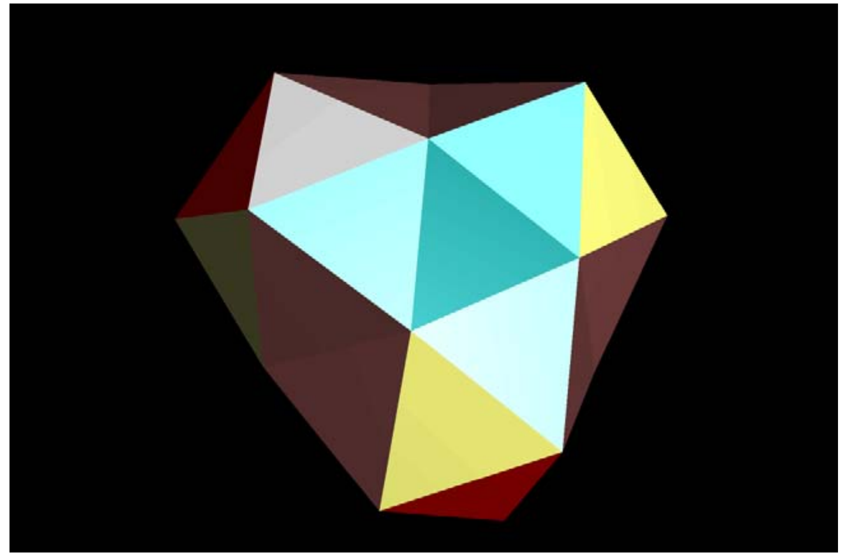
Unfortunately, spheniating and ambiating the convex uniform snub polyhedra does not yield new deltahedra, because the squares and pentagons remain. But the icosahedron, in its guise as the snub tetratetrahedron, yields two new Cundy deltahedra when spheniated at six pairs of equits or ambiated at four patches of equits. One may spheniate and ambiate certain polyhedra repeatedly (though not endlessly), creating a variety of triform, quadriform, quinqueform, and so forth, deltahedra and other acoptic regular-faced polyhedra. Ambiating an icosahedron a second time (at the four unambiated equits) produces a triform deltahedron.

Here it might behoove me to touch upon a few other operations that might have yielded biform acoptic deltahedra but thus far have not. First, we should consider the possibility of appending a set of faces to a core polyhedron at a disjoint set of faces, which would create a "handle" in the surface of the figure. Unfortunately, even if this could be done symmetrically to a uniform core polyhedron, the handle itself requires at



least two different kinds of vertices (one kind above and one kind below), and so would yield at best a triform "ansate" deltahedron. Second, we might augment a face of the core polyhedron and connect the lateral faces of the resulting pyramid directly to adjoining faces of the core with triangles. Unfortunately, in all cases the added triangles are too far from being equits for the "spring model" function to pull them into equits, and no new Cundy deltahedra result. Third, we might begin with a quasi-uniform core, that is, a core polyhedron all of whose vertices are of one kind but whose faces are not all regular polygons. Again this fails to yield new deltahedra; the "spring model" function pulls them into deltahedra that are already in the table. This also happens when we ambiate with appendages other than antiprisms, such as prisms and cupolas. The "spring model" function simply finds the same deltahedra over again.

[23] **Trispheniated octahedron**: Produced by spheniating the three pairs of equits of an octahedron that lie between two opposite faces, this acoptic biform deltahedron has the rotational symmetry group $[2^+,6]$ of an equit antiprism and is chiral. It was absent from Cundy's original table, but was



discovered by Mason Green and Jim McNeill. When the octahedron is thus trispheniated, its vertices shift so that the opposite equits move slightly apart and rotate relative to one another. Calculating the vertex coordinates for this figure involves some pretty tedious algebra, so it is fortunate that Great Stella can perform such calculations effortlessly. Any acoptic *n*-gonal antiprism may be *n*-spheniated to produce a biform acoptic polyhedron, which Green and McNeill have dubbed a **cingulated *n*-gonal antiprism**. (12−30+20)

[24] **Hexaspheniated icosahedron**: Six pairs of equits situated cubically on the icosahedron (these would be the twelve "snub faces" of the icosahedron as a snub tetratetrahedron) may be symmetrically spheniated to produce a nonconvex acoptic biform deltahedron with the ionic (pyritohedral) symmetry group $[3^+,4]$. It was absent from Cundy's list. As with [23], calculating the general coordinates of the vertices involves tedious algebra that Great Stella handles effortlessly. The eight faces of the icosahedron that remain (colored teal in the illustration) are slightly rotated from their original positions and their twelve vertices have a

slightly smaller circumsphere. As far as I know, this is the first published appearance of this polyhedron, in whose Great Stella construction I was assisted by Roger Kaufman. (24–66+44)

[25] **Tetrambiated icosahedron**: Four patches of four equits situated tetrahedrally on the icosahedron may each be symmetrically ambiated with a patch of ten equits. This converts the icosahedron into a Cundy deltahedron with 44 faces (four faces remain from the core icosahedron, colored teal in the illustrations, plus ten from each of the four ambiations). It is chiral, with the tetrahedral rotational symmetry group [3,3], and it was absent from Cundy's original tabulation. As far as I know, this is the first published appearance of this polyhedron; I provide two illustrations of it to better display its shape. (24–66+44)

I believe this tabulation exhausts all the possibilities for nonconvex acoptic biform deltahedra. Together with Freudenthal and van der Waerden's five convex biform deltahedra, we may aver that there are exactly 30 acoptic biform deltahedra. I have made .stel files of all these, but readers who own Great Stella might prefer to build their own (it is quite easy). To build physical models of these figures to the same edge length would require cutting out and pasting together a hoard of equits, all from just one simple template(!).

Cundy's original tabulation included the rhombicuboctahedron augmented with pyramids on all its squares, along with its excavated counterpart. These were #11 and #12, respectively, in his list. It should be readily apparent, however, that these are triform, not biform, deltahedra, since the squares of a rhombicuboctahedron fall into two different symmetry classes.

Still remaining to be enumerated are the biform star-deltahedra, the acoptic triform deltahedra, and the various nonconvex biform polyhedra and star-polyhedra that do not belong to infinite sets with prismatic symmetries. I suspect there are many hundreds of acoptic triform deltahedra, since one may symmetrically augment, excavate, gyraugment, gyrexcavate, spheniate, or ambiate practically any of the 30 biform deltahedra, many in more ways than one. And finally, one result of Freudenthal and van der Waerden's work is to establish that all *n*-form deltahedra for *n*>2 must be nonconvex.

Acknowledgments
My thanks to Branko Grünbaum for supplying references and information about Freudenthal & van der Waerden's and Cundy's papers on deltahedra—and for reminding me that I had overlooked Cundy deltahedron [7] in an earlier version of my table(!); to Roger Kaufman for assistance with Great Stella's "spring model" function—in particular for showing me how to build deltahedron [24] with it; to Lem Chastain for many interesting comments on deltahedra; to Mason Green and Jim McNeill for letting me know about their prior discovery of [23] above; and of course to Robert Webb and his fabulous Great Stella software for building and displaying all kinds of virtual polyhedra (including those illustrated in this article). Alas, I have not had this document peer-reviewed, so any mistakes are mine alone by definition.

Addendum
In reviewing this work, Branko Grünbaum noted (unfortunately, too late for inclusion in the body of this paper) that Freudenthal and van der Waerden were preceded in their account of the five nonregular convex deltahedra by O. Rausenberger's 1915 paper, titled "Konvexe pseudoreguläre Polyeder," in *Zeitschrift für mathematischen und naturwissenschaftlichen Unterricht 46*: 135–142. This is currently the earliest reference to these figures known to me, which mandates revising some of this paper's several references to Freudenthal and van der Waerden's work.

—November 30, 2006